Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 19, 2003*
Accepted: *April 24, 2003*

$N = 2$ supersymmetric scalar-tensor couplings

Ulrich Theis
Institute for Theoretical Physics, Vienna University of Technology
Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria
E-mail: theis@hep.itp.tuwien.ac.at

Stefan Vandoren
Institute for Theoretical Physics and Spinoza Instituut
Utrecht University, 3508 TD Utrecht, The Netherlands
E-mail: S.Vandoren@phys.uu.nl

JHEP04(2003)042

Abstract: We determine the general coupling of a system of scalars and antisymmetric tensors, with at most two derivatives and undeformed gauge transformations, for both rigid and local $N = 2$ supersymmetry in four-dimensional spacetime. Our results cover interactions of hyper, tensor and double-tensor multiplets and apply among others to Calabi-Yau threefold compactifications of type II supergravities. As an example, we give the complete lagrangian and supersymmetry transformation rules of the double-tensor multiplet dual to the universal hypermultiplet.

Keywords: Supersymmetry and Duality, Extended Supersymmetry, Supergravity Models.

Contents

1. Introduction

It is well-known that gauge fields of form degree $D-2$ (antisymmetric tensors) in D dimensions represent one bosonic degree of freedom, and that often this degree of freedom can alternatively be described in terms of a scalar field. The duality transformation relating the two formulations of the same physics exchanges Bianchi identities and field equations. Such tensors arise for instance in compactifications of superstring theories to four dimensions as descendants of the Neveu-Schwarz two-form and the Ramond-Ramond p-forms. Usually one prefers to dualize these two-forms in four dimensions into scalars, in order to understand the full duality group of the theory and to exhibit the geometric structure of the resulting moduli spaces. In some cases, however, it is convenient, or even necessary, to keep the tensors as they naturally appear in string theory. One example is when there are bare gauge potentials, not appearing only through their field strength. This complicates, or might even obstruct, the dualization procedure. Another example is the euclidean formulation of string theory and its compactifications, as used in instanton calculations: The euclidean scalar formulation typically suffers from an indefinite action which invalidates the semiclassical approximation, whereas the dual tensor formulation has a positive semi-definite action. Such a situation occurs for instance for the ten-dimensional D-instanton in type IIB string theory [1], but also for instantons that contribute to the universal hypermultiplet effective action in four dimensions [2]. Therefore, it is necessary to have a better understanding of general scalar-tensor couplings, and the aim of this paper is to construct their lagrangians and transformation rules with $N=2$ supersymmetry in four spacetime dimensions.

This paper was motivated by Calabi-Yau threefold compactifications of type II string theories, which yield matter-coupled $N=2$ supergravity as the low energy effective theory. In the absence of internal fluxes, the vector multiplets arising from the compactification do not couple to the other matter fields and can consistently be truncated. For type IIA, there remain $h_{1,2}$ hypermultiplets and one tensor multiplet (containing three scalars and one tensor). For type IIB, one has $h_{1,1}$ tensor multiplets and one double-tensor multiplet (containing two scalars and two tensors). The tensor multiplet in IIA and the double-tensor multiplet in IIB are universal, in that they appear for all choices of the Calabi-Yau manifold and do not depend on its moduli. The generic situation is that one gets a complicated four-dimensional low energy supergravity action with interacting scalars and tensors. Superspace effective actions for type II strings were derived in [3], and we are interested in the on-shell component formulation that include also the double-tensor multiplet. At tree level, the bosonic terms of type IIA and type IIB supergravity on a Calabi-Yau were determined in [4] and [5], respectively, but the fermionic terms and the supersymmetry transformations are still lacking. While the general coupling of an arbitrary number of hypermultiplets to $N = 2$ supergravity was determined in [6], for tensor or double-tensor multiplets a similar program has not been carried out so far. To our knowledge, the latter has in fact never been fully coupled to supergravity before, while at least locally supersymmetric versions of tensor multiplets (in the superconformal approach) have appeared in the literature in [7, 3, 8]. In this article we close these gaps. We do this by employing the duality to hypermultiplets: The general $N = 2$ supersymmetric system of n_T tensors and a number[1] of scalars, where the former are subject to standard field-independent gauge transformations, is equivalent to a set of hypermultiplets. Their scalars will parametrize a target space whose metric has n_T commuting isometries (in appropriate coordinates these are just the Peccei-Quinn shift symmetries). Hence, the general $N = 2$ supersymmetric coupling of tensors to scalars can be derived by dualizing the general system of interacting hypermultiplets with a number of abelian isometries.

Our insistence on undeformed gauge transformations ensures that the tensors occur only through their field strengths.[2] At least for rigid supersymmetry, this does not yield the most general self-interactions of tensors. There are in addition Freedman-Townsend models that involve bare gauge potentials,[3] but that are still dual to nonlinear sigma models [9]. The distinguishing feature of Freedman-Townsend couplings is that the gauge transformations of the tensors are deformed and become field-dependent. As was argued in [10], there seem to exist globally supersymmetric self-interactions of the double-tensor multiplet that are of this form. We exclude such models from our consideration, but this does not significantly reduce the applicability of our results, since we are mainly interested in Calabi-Yau compactifications of type II strings, where such Freedman-Townsend couplings do not appear.

The paper is organized as follows: In section 2 we first give a brief review of globally supersymmetric hypermultiplet interactions and discuss commuting isometries necessary for the dualization of scalars into tensors. We then explain in detail the dualization procedure via the gauging of isometries and the derivation of the supersymmetry transformations for

[1] This number is not arbitrary but restricted by supersymmetry to be a multiple of four minus n_T.

[2] There are no Chern-Simons terms for 2-forms in four dimensions.

[3] To first order in the coupling constant, the interactions are of the form $gf^{IJK}B_I *\mathrm{d}B_J *\mathrm{d}B_K$, where f^{IJK} are the structure constants of some Lie algebra.

the dual theory. The reader not interested in this derivation can go directly to the last subsection, where the scalar-tensor multiplet couplings are directly given, without referring to the dual hypermultiplet system. In section 3 we repeat these steps (in less detail) for local supersymmetry, and in section 4 we discuss the example of the universal hypermultiplet and its dual formulation in terms of the double-tensor multiplet.

2. Rigid supersymmetry

As mentioned in the introduction, our strategy is to start with hypermultiplet actions with a number of abelian isometries, and to derive the action for the scalar-tensor system by dualizing with respect to these isometries. This method produces the most general action for scalars coupled to tensors, where the tensors only appear through their field strength. First, we briefly review the hypermultiplet lagrangian, its supersymmetry rules, and the geometry of the target space manifold.

2.1 Hypermultiplets

The lagrangian for n hypermultiplets contains $4n$ real scalars $\phi^{\hat{A}}$ and $2n$ two-component spinors λ^a. Throughout this paper, we focus on four spacetime dimensions. It is well-known [11, 6, 12] that for rigid $N = 2$ supersymmetry, the scalar fields $\phi^{\hat{A}}$ parametrize a hyperkähler manifold. Such manifolds have holonomy group contained in $\mathrm{Sp}(n)$, and we denote the metric by $G_{\hat{A}\hat{B}}(\phi)$. Our discussion closely follows [12, 13], but we change to Wess & Bagger conventions [14]. The general form of the lagrangian can then be written as

$$\mathcal{L}_{\mathrm{H}} = -\frac{1}{2} G_{\hat{A}\hat{B}}\, \partial^\mu \phi^{\hat{A}}\, \partial_\mu \phi^{\hat{B}} - \frac{\mathrm{i}}{2} h_{a\bar{a}} \left(\lambda^a \sigma^\mu \overleftrightarrow{D}_\mu \bar{\lambda}^{\bar{a}} \right) + \frac{1}{4} W_{ab\,\bar{a}\bar{b}}\, \lambda^a \lambda^b\, \bar{\lambda}^{\bar{a}} \bar{\lambda}^{\bar{b}} \,, \tag{2.1}$$

where $\bar{\lambda}^{\bar{a}} = (\lambda^a)^*$, and $h_{a\bar{a}}(\phi)$ and $W_{ab\,\bar{a}\bar{b}}(\phi)$ are field-dependent tensors. The covariant derivative contains a connection $\Omega_{\hat{A}}{}^a{}_b$,

$$D_\mu \lambda^a = \partial_\mu \lambda^a + \partial_\mu \phi^{\hat{A}}\, \Omega_{\hat{A}}{}^a{}_b\, \lambda^b \,, \tag{2.2}$$

such that the lagrangian is subject to two kinds of equivalence relations, those associated with target space diffeomorphisms $\phi \to \phi'(\phi)$, and those associated with reparametrizations of the fermion frame $\lambda^a \to S^a{}_b(\phi)\, \lambda^b$, and similarly for other quantities carrying $a, \bar{b}$ indices. These equivalence relations later allow us to use a convenient basis in which the dualization procedure can be easily carried out.

The rigid supersymmetry transformations for the bosons are parametrized by (inverse) vielbeins $\gamma^{\hat{A}}_{ia}(\phi)$ such that

$$\delta_\epsilon \phi^{\hat{A}} = \gamma^{\hat{A}}_{ia}\, \epsilon^i \lambda^a + \bar{\gamma}^{i\hat{A}}_{\bar{a}}\, \bar{\epsilon}_i\, \bar{\lambda}^{\bar{a}} \,. \tag{2.3}$$

Under complex conjugation, we have that $\bar{\epsilon}_i = (\epsilon^i)^*$, with $i = 1, 2$. The fermion transformations must be covariant with respect to the redefinitions of the fermion frame. Hence we parametrize

$$\delta_\epsilon \lambda^a + \delta_\epsilon \phi^{\hat{A}}\, \Omega_{\hat{A}}{}^a{}_b\, \lambda^b = \mathrm{i}\, \partial_\mu \phi^{\hat{A}}\, V^{ai}_{\hat{A}}\, \sigma^\mu \bar{\epsilon}_i \,, \tag{2.4}$$

for some quantities $V^{ai}_{\hat{A}}(\phi)$. The transformation on $\bar{\lambda}^{\bar{a}}$ follows by complex conjugation.

The closure of the supersymmetry algebra and the supersymmetry invariance of the action imposes constraints on the various quantities which enforce the geometry of the target space to be hyperkähler. This was worked out in detail in [12, 13], and we summarize it here. First of all, the vielbeins and the tensor $h_{a\bar{a}}$ are covariantly constant with respect to the Levi-Civita and Sp(n) connections. Furthermore, one finds

$$\gamma^{\hat{A}}_{ia}\, V^{aj}_{\hat{B}} + \bar{\gamma}^{j\hat{A}}{}_{\bar{a}}\, \bar{V}^{\bar{a}}_{\hat{B}i} = \delta^j_i\, \delta^{\hat{A}}_{\hat{B}}\,, \qquad \gamma^{\hat{A}}_{ib}\, V^{aj}_{\hat{A}} = \delta^j_i\, \delta^a_b\,, \qquad \bar{V}^{\bar{a}}_{\hat{A}(i}\, \gamma^{\hat{A}}_{j)b} = 0\,. \tag{2.5}$$

Moreover, there is a relation between these vielbeins, the metric G and the tensor h,

$$G_{\hat{A}\hat{B}} = h_{a\bar{b}}\, V^{ai}_{\hat{A}}\, \bar{V}^{\bar{b}}_{\hat{B}i}\,, \qquad h_{a\bar{b}} = \frac{1}{2}\, G_{\hat{A}\hat{B}}\, \gamma^{\hat{A}}_{ia}\, \bar{\gamma}^{i\hat{B}}{}_{\bar{b}}\,. \tag{2.6}$$

From this, one derives furthermore that

$$\gamma^{\hat{A}}_{ia}\, V^{ai}_{\hat{B}} = \delta^{\hat{A}}_{\hat{B}}\,. \tag{2.7}$$

Hyperkähler manifolds have three covariantly constant complex structures $\vec{J}$ that satisfy the quaternionic algebra. In our notation, they are

$$\vec{J}_{\hat{A}\hat{B}} = \mathrm{i} h_{b\bar{a}}\, \bar{V}^{\bar{a}}_{\hat{A}i}\, \vec{\tau}^{\,i}{}_j V^{bj}_{\hat{B}}\,, \tag{2.8}$$

where $\vec{\tau}$ are the three Pauli matrices.

Finally, there are the curvature relations

$$W_{ab\bar{a}\bar{b}} = -\frac{1}{2}\, h_{c\bar{a}}\, \gamma^{\hat{A}}_{ia}\, \bar{\gamma}^{i\hat{B}}{}_{\bar{b}}\, R_{\hat{A}\hat{B}}{}^c{}_b = -\frac{1}{4}\, \gamma^{\hat{A}}_{ia}\, \bar{\gamma}^{i\hat{B}}{}_{\bar{b}}\, \gamma^{\hat{C}}_{jb}\, \bar{\gamma}^{j\hat{D}}{}_{\bar{a}}\, R_{\hat{A}\hat{B}\hat{C}\hat{D}}\,, \tag{2.9}$$

where we have defined $R_{\hat{A}\hat{B}} = 2(\partial_{[\hat{A}} + \Omega_{[\hat{A}})\, \Omega_{\hat{B}]}$. This curvature takes values in $sp(n)$ because it commutes with the antisymmetric and covariantly constant tensor

$$\mathcal{E}_{ab} = \frac{1}{2}\varepsilon^{ji}\, G_{\hat{A}\hat{B}}\, \gamma^{\hat{A}}_{ia}\, \gamma^{\hat{B}}_{jb}\,. \tag{2.10}$$

Combined with the inverse $h^{\bar{a}b}$, one can change barred indices into unbarred ones and vice versa. One can then show that the four-fermi tensor W_{abcd} is completely symmetric in its four indices.

2.2 Commuting isometries

For a bosonic sigma model, one can dualize scalars into tensors if they appear in the lagrangian only through their derivatives. In a coordinate invariant setting, this means that the target space has a set of abelian Killing vectors,

$$\delta_\theta \phi^{\hat{A}} = \theta^I k^{\hat{A}}_I(\phi)\,, \qquad [k_I\,,\, k_J] = 0\,. \tag{2.11}$$

Using Frobenius' theorem, one can then choose coordinates $\{\phi^I, \phi^A\}$ such that these transformations act as constant shifts on ϕ^I while leaving ϕ^A invariant,

$$\delta_\theta \phi^I = \theta^I\,, \qquad \delta_\theta \phi^A = 0\,, \qquad I = 1, \ldots, n_T\,, \qquad A = 1, \ldots, 4n - n_T\,, \tag{2.12}$$

JHEP04(2003)042

and the lagrangian depends on the former only through their field strengths $\partial_\mu \phi^I$. Vice versa, tensors can be dualized into scalars if (but not only if) they appear only through their field strengths, yielding commuting isometries in the corresponding sigma model. This class of scalar-tensor models is the one we are interested in, and within this class our dualization procedure is general.[4]

In an $N=2$ supersymmetric sigma model, dualization can only be done if the target space isometries are triholomorphic [15]. This means that the Killing vectors leave the complex structures invariant, and that the isometries commute with supersymmetry. Generically this implies that the fermions transform non-trivially, as was worked out in detail in [13],

$$\delta_\theta \lambda^a + \delta_\theta \phi^{\hat{A}}\, \Omega_{\hat{A}}{}^a{}_b \lambda^b = \theta^I t_I{}^a{}_b \lambda^b\,, \tag{2.13}$$

with matrices

$$t_I{}^a{}_b(\phi) = \frac{1}{2}\, V^{ai}_{\hat{B}}\, \gamma^{\hat{A}}_{ib}\, D_{\hat{A}} k^{\hat{B}}_I\,. \tag{2.14}$$

For riemannian hyperkähler manifolds of real dimension $4n$, the holonomy group is contained in $\mathrm{Sp}(n)$, and the group of triholomorphic isometries must be a subgroup thereoff. The maximum number of commuting triholomorphic isometries is therefore equal to the rank n.[5] If we dualize $n_T \leq n$ scalars, we obtain a model with $4n - n_T$ scalars coupled to n_T tensors. The case of $n_T = n$ yields self-interacting $N=2$ tensor multiplets and was studied in [15] using superspace techniques.

Using the equivalence relation based on the transformation $\lambda^a \to S^a{}_b(\phi)\lambda^b$, one can choose an $\mathrm{Sp}(n)$ frame such that the fermions do not transform. In the basis (2.12), this requires

$$\partial_I\, S^a{}_b = -S^a{}_c (t_I - \Omega_I)^c{}_b\,. \tag{2.15}$$

A solution can only be found if the integrability conditions are satisfied. This is indeed the case, as one can check, due to the general identity

$$D_{\hat{A}} t_I{}^a{}_b = k^{\hat{B}}_I\, R_{\hat{A}\hat{B}}{}^a{}_b\,. \tag{2.16}$$

Similarly, the same transformation puts the vielbeins $\gamma^{\hat{A}}_{ia}$ and $V^{ai}_{\hat{A}}$ in a basis where they do not depend on the scalar fields ϕ^I, so we conclude that

$$\delta_\theta\, \lambda^a = 0\,, \qquad \delta_\theta \gamma^{\hat{A}}_{ia} = 0\,. \tag{2.17}$$

Through the covariant constancy of the vielbeins, $D_{\hat{B}} \gamma^{\hat{A}}_{ia} = 0$, it follows that also the $\mathrm{Sp}(n)$ connection is independent of ϕ^I, and, using (2.14), the Ith component is equal to

$$\Omega_I{}^a{}_b = \frac{1}{2}\, \Gamma_{I\hat{A}}{}^{\hat{B}}\, V^{ai}_{\hat{B}}\, \gamma^{\hat{A}}_{ib} = t_I{}^a{}_b\,. \tag{2.18}$$

[4]We may assume that G_{IJ} is invertible.

[5]For pseudo-riemannian hyperkähler manifolds, the holonomy group is generically non-compact, and there can be more than n triholomorphic abelian isometries. These cases are relevant in the context of the superconformal tensor calculus, and examples were given in [16].

JHEP04(2003)042

Hence the complete hypermultiplet lagrangian (2.1) is invariant. Notice that there are still residual transformations $\lambda^a \to S^a{}_b(\phi^A)\lambda^b$ that define equivalence classes, since (2.15) is now trivially satisfied.

The conserved Noether currents of the shift symmetries (2.12), (2.17) are

$$J^\mu_I = G_{I\hat{A}}\partial^\mu\phi^{\hat{A}} - \mathrm{i}h_{b\bar{a}}\,\Omega_I{}^b{}_a\,\lambda^a\sigma^\mu\bar{\lambda}^{\bar{a}}\,; \tag{2.19}$$

its divergence is given by the field equations of ϕ^I,

$$\partial_\mu J^\mu_I = \frac{\delta S_{\mathrm{H}}}{\delta\phi^I}\,. \tag{2.20}$$

In deriving (2.19), we have used covariant constancy of $h_{a\bar{a}}$ and $\partial_I h_{a\bar{a}} = 0$. Combined with the covariant constancy of (2.10), this leads to the relations

$$\Omega_I{}^b{}_a h_{b\bar{a}} + \bar{\Omega}_I{}^{\bar{b}}{}_{\bar{a}} h_{a\bar{b}} = 0\,, \qquad \Omega_I{}^c{}_{[a}\,\mathcal{E}_{b]c} = 0\,, \tag{2.21}$$

which imply that the matrices Ω_I must be contained in $sp(n)$.

2.3 Dualization

The dualization procedure can geometrically be understood as follows [17, 18]: one gauges the isometries and adds Lagrange multipliers that constrain the field strengths of the gauge potentials to be trivial. Integrating out the multipliers sets the gauge fields to zero,[6] giving back the original action, whereas integrating out the gauge potentials yields the dual action.

For an $N = 2$ supersymmetric sigma model, the gauging is done by minimal coupling to (in our case abelian) $N = 2$ vector multiplets. Since these multiplets are not propagating and just serving as a background, we can consistently freeze the other fields of the vector multiplets (the gauginos, scalars and auxiliary fields) to zero. We therefore only replace the derivatives of ϕ^I with covariant derivatives,

$$\partial_\mu\phi^I \to \partial_\mu\phi^I - A^I_\mu\,, \tag{2.22}$$

such that to linear order the gauge fields couple to the currents J^μ_I obtained in (2.19). To maintain manifest $N = 2$ supersymmetry, one has to add extra terms to the lagrangian proportional to the other fields of the vector multiplets. More details about these terms are discussed below and can be found explicitly in [13], or in earlier papers on the subject. In our chosen background they vanish, but their variations do not. For instance, the doublets of gaugino fields χ^{Ii} vary into the field strengths according to

$$\delta_\epsilon\chi^{Ii} = \sigma^{\mu\nu}\epsilon^i F^I_{\mu\nu}\,. \tag{2.23}$$

We now introduce multipliers $B_{\mu\nu I}$ and consider the lagrangian

$$\mathcal{L} = \hat{\mathcal{L}}_{\mathrm{H}} - H^\mu_I\,A^I_\mu\,, \tag{2.24}$$

[6] Actually, it restricts them to be pure gauge, but the gauge parameters can be absorbed into the scalars by a field redefinition.

where

$$H_I^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma I} \tag{2.25}$$

are the dual field strengths of the tensors, and where $\hat{\mathcal{L}}_{\rm H}$ stands for the lagrangian (2.1) with all derivatives of ϕ^I made covariant. It is equal to the totally gauged and supersymmetric action, where the other fields of the vector multiplets are set to zero. $\mathcal{L}$ is invariant (up to a total derivative) under gauged isometries thanks to the Bianchi identities of H_I^μ.

Substituting the solution to the field equations for the gauge fields A_μ^I,

$$G_{IJ}A^{\mu J} = J_I^\mu - H_I^\mu\,, \tag{2.26}$$

into $\mathcal{L}$ yields, up to a surface term $\mathcal{L}_{\rm surf} = -\partial_\mu\phi^I H_I^\mu$, the scalar-tensor lagrangian

$$\begin{aligned}\mathcal{L}_{\rm T} = \frac{1}{2}\,M^{IJ}H_I^\mu H_{\mu J} - \frac{1}{2}\,\mathcal{G}_{AB}\,\partial^\mu\phi^A\,\partial_\mu\phi^B - A_A^I\,H_I^\mu\,\partial_\mu\phi^A - \frac{\rm i}{2}\,h_{a\bar a}\left(\lambda^a\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\lambda^{\bar a}\right) + \\ + {\rm i}h_{a\bar a}\,H_{\mu I}M^{IJ}\Omega_J{}^a{}_b\,\lambda^b\sigma^\mu\bar\lambda^{\bar a} + \frac{1}{4}\,V_{ab\,\bar a\bar b}\,\lambda^a\lambda^b\,\bar\lambda^{\bar a}\bar\lambda^{\bar b}\,.\end{aligned} \tag{2.27}$$

The tensors appearing in the kinetic terms for the bosonic fields are

$$M^{IJ} = (G_{IJ})^{-1}\,, \qquad \mathcal{G}_{AB} = G_{AB} - G_{AI}M^{IJ}G_{JB}\,, \qquad A_A^I = M^{IJ}G_{JA}\,. \tag{2.28}$$

Note that due to the isometries $\delta_\theta\phi^I = \theta^I$, upon dualization the hypermultiplet metric $G_{\hat A\hat B}$ splits up similarly to a dimensional reduction on a torus:

$$G_{\hat A\hat B}\,{\rm d}\phi^{\hat A}\otimes{\rm d}\phi^{\hat B} = \mathcal{G}_{AB}\,{\rm d}\phi^A\otimes{\rm d}\phi^B + G_{IJ}\left({\rm d}\phi^I + A_A^I\,{\rm d}\phi^A\right)\otimes\left({\rm d}\phi^J + A_B^J\,{\rm d}\phi^B\right). \tag{2.29}$$

For the maximal number of commuting triholomorphic isometries, i.e. $n_T = n$, these metrics were studied in [15, 19]. The A_A^I are target space vector fields with abelian gauge transformations $\delta_\Lambda A_A^I = \partial_A\,\Lambda^I(\phi^B)$, which leave the lagrangian (2.27) invariant (modulo a total derivative).

Furthermore, the covariant derivative of λ^a is given by

$$\mathcal{D}_\mu\lambda^a = \partial_\mu\lambda^a + \partial_\mu\phi^A\,\Gamma_A{}^a{}_b\,\lambda^b\,, \tag{2.30}$$

with connection

$$\Gamma_A{}^a{}_b = \Omega_A{}^a{}_b - A_A^I\Omega_I{}^a{}_b\,. \tag{2.31}$$

$\Gamma_A{}^a{}_b$ is invariant under the gauge transformations generated by δ_Λ and transforms under redefinitions of the fermion frame like $\Omega_A{}^a{}_b$.

Finally, for the four-fermi term in the lagrangian, we have introduced

$$V_{ab\,\bar a\bar b} = W_{ab\,\bar a\bar b} + 4h_{c\bar a}\,\Omega_I{}^c{}_{(a}M^{IJ}\Omega_J{}^d{}_{b)}\,h_{d\bar b}\,. \tag{2.32}$$

After conversion of the barred indices into unbarred ones by contracting with $h^{\bar a d}\mathcal{E}_{dc}$, we observe that V_{abcd} is still completely symmetric.

The supersymmetry transformations for the tensor fields leaving $S_{\rm T}$ invariant can be deduced from requiring that (2.24) leads to an invariant action. Therefore we first need to know how $\hat{\mathcal{L}}_{\rm H}$ transforms. As mentioned above, this lagrangian is the part of the fully gauged and supersymmetric action where the other fields of the vector multiplets are set to zero. This action can be written as $S_{\rm HV} = \hat{S}_{\rm H} + \Delta S$, and the only term in ΔS relevant for our considerations is the one containing the gauginos,

$$\Delta\mathcal{L} = h_{a\bar{a}} \bar{V}^{\bar{a}}_{Ii}\, \lambda^a \chi^{Ii} + \text{c.c.} + \dots . \tag{2.33}$$

The dots indicate terms proportional to the vector multiplet scalars or auxiliary fields, whose supersymmetry variations vanish in our chosen background. From invariance of $S_{\rm HV}$ and using (2.23) we derive that (modulo a total derivative)

$$\delta_\epsilon \hat{\mathcal{L}}_{\rm H} = h_{a\bar{a}} \bar{V}^{\bar{a}}_{Ii}\, \epsilon^i \sigma^{\mu\nu} \lambda^a\, F^I_{\mu\nu} + \text{c.c.} \tag{2.34}$$

It is now easy to write down the compensating transformation for the tensors,

$$\delta_\epsilon B_{\mu\nu I} = 2\mathrm{i}\, h_{a\bar{a}} \bar{V}^{\bar{a}}_{Ii}\, \epsilon^i \sigma_{\mu\nu} \lambda^a + \text{c.c.} \tag{2.35}$$

The supersymmetry transformations of the fermions λ^a in the dual formulation are derived by making the replacement (2.22) in (2.4) (further corrections to the λ^a transformations due to coupling to the vector multiplets involve fields that vanish in our background) and then substituting the solution (2.26) for A^I_μ. This gives

$$\begin{aligned} \delta_\epsilon \lambda^a = {} & \mathrm{i}\left[\partial_\mu \phi^A \left(V^{ai}_A - V^{ai}_I M^{IJ} G_{JA}\right) + H_{\mu I} M^{IJ} V^{ai}_J\right] \sigma^\mu \bar{\epsilon}_i - \\ & - \left[\delta_\epsilon \phi^{\hat{A}}\, \Omega_{\hat{A}}{}^a{}_b - 2 M^{IJ} V^{ai}_I h_{c\bar{b}}\, \Omega_J{}^c{}_b\, \bar{\epsilon}_i \bar{\lambda}^{\bar{b}}\right] \lambda^b . \end{aligned} \tag{2.36}$$

The transformations of the scalars ϕ^A remain the same. Notice that this formula still contains terms proportional to $\delta_\epsilon \phi^I$ and therefore γ^I_{ia}, which are quantities that should not appear after dualization. Using (2.5) and (2.6), one can however express γ^I_{ia} in terms of other known quantities on the scalar-tensor multiplet side,

$$\gamma^I_{ia} = M^{IJ} h_{a\bar{b}} \bar{V}^{\bar{b}}_{Ji} - \gamma^A_{ia} A^I_A . \tag{2.37}$$

By construction, the algebra of the supersymmetry transformations thus derived closes on-shell (modulo gauge transformations of the tensors, see below), since they are symmetries of the action $S_{\rm T}$. This completes the dualization procedure.

2.4 Scalar-tensor multiplets

With the insight of the previous section, we can now formulate the scalar-tensor multiplet lagrangian and supersymmetry rules without referring to hypermultiplets. The $N = 2$ scalar-tensor system consists of n_T tensors $B_{\mu\nu I}$ and $4n - n_T$ scalars ϕ^A, together with $2n$ two-component spinors λ^a.

The supersymmetry transformation rules are parametrized as

$$\begin{aligned} \delta_\epsilon \phi^A &= \gamma^A_{ia}\, \epsilon^i \lambda^a + \bar{\gamma}^{iA}{}_{\bar{a}}\, \bar{\epsilon}_i\, \bar{\lambda}^{\bar{a}} \\ \delta_\epsilon B_{\mu\nu I} &= 2\mathrm{i}\, g_{Iia}\, \epsilon^i \sigma_{\mu\nu} \lambda^a - 2\mathrm{i}\, \bar{g}^i_{I\bar{a}}\, \bar{\epsilon}_i\, \bar{\sigma}_{\mu\nu} \bar{\lambda}^{\bar{a}} \\ \delta_\epsilon \lambda^a &= \mathrm{i}\partial_\mu \phi^A\, W^{ai}_A\, \sigma^\mu \bar{\epsilon}_i + \mathrm{i}\left(H^\mu_I + k_{Ib\bar{b}}\, \lambda^b \sigma^\mu \bar{\lambda}^{\bar{b}}\right) f^{Iai}\, \sigma_\mu \bar{\epsilon}_i - \\ & \quad - \delta_\epsilon \phi^A\, \Gamma_A{}^a{}_b\, \lambda^b - \left(g_{Iic}\, \epsilon^i \lambda^c + \bar{g}^i_{I\bar{c}}\, \bar{\epsilon}_i \bar{\lambda}^{\bar{c}}\right) \Gamma^{Ia}{}_b\, \lambda^b , \end{aligned} \tag{2.38}$$

for some unknown quantities γ^A_{ia}, g_{Iia} etc. This is not the most general Ansatz possible, but on-shell, and in the presence of an action,[7] it is sufficient since all the coefficient functions appearing here are related by dualization to hypermultiplet quantities. Comparing with (2.35) and (2.36) and using (2.31), we find

$$\begin{aligned} g_{Iia} &= h_{a\bar a}\bar V^{\bar a}_{Ii} = G_{I\hat A}\gamma^{\hat A}_{ia}\,, \qquad f^{Iai} = M^{IJ}V^{ai}_J\,, \\ W^{ai}_A &= V^{ai}_A - A^I_A V^{ai}_I\,, \qquad \Gamma^{Ia}{}_b = M^{IJ}\Omega_J{}^a{}_b\,, \qquad k_{Ib\bar b} = \mathrm{i}h_{c\bar b}\,\Omega_I{}^c{}_b\,. \end{aligned} \tag{2.39}$$

If we would consider a more general Ansatz, we would find that the extra quantities would have to vanish, or would be equivalent to the Ansatz (2.38).

The action is parametrized by

$$\begin{aligned} \mathcal{L}_{\mathrm{T}} = &\frac{1}{2}M^{IJ}H^\mu_I H_{\mu J} - \frac{1}{2}\mathcal{G}_{AB}\,\partial^\mu\phi^A\,\partial_\mu\phi^B - A^I_A\,H^\mu_I\,\partial_\mu\phi^A - \frac{\mathrm{i}}{2}h_{a\bar a}\left(\lambda^a\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\lambda^{\bar a}\right) + \\ &+H_{\mu I}M^{IJ}k_{Ja\bar a}\,\lambda^a\sigma^\mu\bar\lambda^{\bar a} + \frac{1}{4}V_{ab\bar a\bar b}\,\lambda^a\lambda^b\,\bar\lambda^{\bar a}\bar\lambda^{\bar b}\,, \end{aligned} \tag{2.40}$$

for some unknown functions M^{IJ}, $\mathcal{G}_{AB}$, etc. We denote $H^\mu_I = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma I}$, and the covariant derivative is given by

$$\mathcal{D}_\mu\lambda^a = \partial_\mu\lambda^a + \partial_\mu\phi^A\,\Gamma_A{}^a{}_b\,\lambda^b\,. \tag{2.41}$$

The connection ensures covariance with respect to fermion frame reparametrizations $\lambda^a \to S^a{}_b(\phi)\lambda^b$.

We now require closure of the supersymmetry algebra and invariance of the action. This imposes constraints on and relations between the various quantities appearing in the action and supersymmetry transformation rules, which must be equivalent with the ones that appear on the hypermultiplet side.

For the commutator of two supersymmetries to give a translation, we find

$$\begin{aligned} \gamma^A_{ia}\,W^{bj}_A + g_{Iia}\,f^{Ibj} &= \delta^j_i\,\delta^b_a \\ \gamma^A_{ia}\,\bar W^{\bar a}_{Aj} + g_{Iia}\,\bar f^{I\bar a}{}_j + (i\leftrightarrow j) &= 0\,, \end{aligned} \tag{2.42}$$

for contractions over A and I, and

$$\begin{pmatrix} \gamma^A_{ia}\,W^{aj}_B & \gamma^A_{ia}\,f^{Jaj} \\ g_{Iia}\,W^{aj}_B & g_{Iia}\,f^{Jaj} \end{pmatrix} + \text{c.c.}(i\leftrightarrow j) = \delta^j_i\begin{pmatrix} \delta^A_B & 0 \\ 0 & \delta^J_I \end{pmatrix}. \tag{2.43}$$

There are further requirements coming from invariance of the action; from the variations proportional to $\lambda\partial^2\phi$ and $\lambda\partial_\mu H_\nu$, we find

$$\mathcal{G}_{AB}\,\gamma^B_{ia} = h_{a\bar b}\,\bar W^{\bar b}_{Ai}\,, \qquad M^{IJ}g_{Jia} = h_{a\bar a}\bar f^{I\bar a}{}_i\,. \tag{2.44}$$

[7]In principle, we could relax the condition that an action exists, in the same spirit as in [20]. In such a setup, it is not clear that (2.38) is general enough.

These relations imply among others that

$$\begin{aligned}
\mathcal{G}_{AB} &= h_{a\bar b}\, W^{ai}_A\, \bar W^{\bar b}_{Bi}\,, \qquad M^{IJ} = h_{a\bar b}\, f^{Iai}\bar f^{J\bar b}{}_i \\
\delta^j_i\, h_{a\bar b} &= \mathcal{G}_{AB}\,\gamma^A_{ia}\,\bar\gamma^{jB}_{\bar b} + M^{IJ} g_{Iia}\,\bar g^j_{J\bar b} \\
\mathcal{E}_{ab} &= \frac{1}{2}\varepsilon^{ji}\left(\mathcal{G}_{AB}\,\gamma^A_{ia}\,\gamma^B_{jb} + M^{IJ} g_{Iia}\, g_{Jjb}\right).
\end{aligned} \tag{2.45}$$

Notice that as a consequence, similar to (2.7),

$$\begin{pmatrix} \gamma^A_{ia}\, W^{ai}_B & \gamma^A_{ia}\, f^{Jai} \\ g_{Iia}\, W^{ai}_B & g_{Iia}\, f^{Jai} \end{pmatrix} = \begin{pmatrix} \delta^A_B & 0 \\ 0 & \delta^J_I \end{pmatrix}. \tag{2.46}$$

All the above relations can alternatively be derived by decomposing their hypermultiplet counterparts according to the dictionary (2.28) and (2.39).

Variations proportional to $\lambda\partial_\mu\phi^A\partial_\nu\phi^B$ vanish if

$$\mathcal{D}_A\, W^{ai}_B = -\frac{1}{2}F_{AB}{}^I\, \bar g^i_{I\bar a}\, h^{\bar a a}\,. \tag{2.47}$$

This determines the field strength $F_{AB}{}^I = 2\partial_{[A}A^I_{B]}$ in terms of other quantities. The covariant derivative $\mathcal{D}_A$ contains connections $\Gamma_A{}^a{}_b$ and the Christoffel symbols $\Gamma_{AB}{}^C$, built from $\mathcal{G}_{AB}$. Note that the right-hand side is antisymmetric in A, B, implying that $\mathcal{D}_{(A}W^{ai}_{B)} = 0$, and it can be interpreted as the torsion tensor of the target space connection.

Variations proportional to $\lambda H_\mu\,\partial_\nu\phi$ now vanish provided that

$$\begin{aligned}
g_{Jia}\partial_A M^{IJ} - \mathcal{G}_{AB}\gamma^B_{ib}\,\Gamma^{Ib}{}_a - h_{a\bar a}\,\mathcal{D}_A\bar f^{I\bar a}{}_i &= 0 \\
F_{AB}{}^I\gamma^B_{ia} + \mathcal{G}_{AB}\gamma^B_{ib}\,\Gamma^{Ib}{}_a - h_{a\bar a}\,\mathcal{D}_A\bar f^{I\bar a}{}_i &= 0\,,
\end{aligned} \tag{2.48}$$

while variations proportional to $\lambda H_\mu H_\nu$ require

$$\gamma^A_{ia}\,\partial_A M^{IJ} = -2M^{IK} g_{Kib}\Gamma^{Jb}{}_a\,. \tag{2.49}$$

Various other relations can of course be derived from this. For instance, one may express $\Gamma^{Ia}{}_b$ in terms of other quantities. Alternatively, one could write the field strength as $F_{AB}{}^I = -2\,h_{b\bar a}\,\bar W^{\bar a}_{Ai}\,\Gamma^{Ib}{}_a\,W^{ai}_B$.

Further consequences are used for the closure of the supersymmetry commutator on the bosons, proportional to fermion bilinears. They can be written as expressions for the covariant derivatives,

$$\mathcal{D}_A\gamma^B_{ia} = -\frac{1}{2}F_{AC}{}^I\,\mathcal{G}^{CB} g_{Iia}\,, \qquad \mathcal{D}_A\, g_{Iia} = -M_{IJ}\,\mathcal{G}_{AB}\gamma^B_{ib}\,\Gamma^{Jb}{}_a\,, \tag{2.50}$$

with $M_{IJ}M^{JK} = \delta^K_I$. Furthermore, we find

$$k_{Ia\bar b} = \mathrm{i}h_{c\bar b}\, M_{IJ}\,\Gamma^{Jc}{}_a\,. \tag{2.51}$$

All these relations are consistent with the case when the tensors decouple from the scalars. This happens when M^{IJ} is constant and the field strength $F_{AB}{}^I$ vanishes.

JHEP04(2003)042

Using (2.48) and (2.50), it follows that $h_{a\bar{b}}$ and $\mathcal{E}_{ab}$ are covariantly constant with respect to the connection $\Gamma_A{}^a{}_b$. This implies that its curvature $\mathcal{R}_{AB} = 2(\partial_{[A} + \Gamma_{[A})\,\Gamma_{B]}$ takes values in $sp(n)$,

$$\mathcal{R}_{AB}{}^c{}_{[a}\,\mathcal{E}_{b]c} = 0\,, \qquad \mathcal{R}_{AB}{}^{\bar{a}}{}_{\bar{b}} = -h_{c\bar{b}}\,h^{\bar{a}d}\,\mathcal{R}_{AB}{}^c{}_d\,. \tag{2.52}$$

To understand the implications of this on the holonomy of the target space, we need to know how the Riemann curvature decomposes into its different components. Taking a second covariant derivative of (2.47) and antisymmetrizing, we find, using (2.46),

$$\mathcal{R}_{ABCD} = h_{a\bar{a}}\,\mathcal{R}_{AB}{}^a{}_b\,W_C^{bi}\,\bar{W}_{Di}^{\bar{a}} + \frac{1}{2}M_{IJ}\,F_{C[A}{}^I F_{B]D}{}^J\,. \tag{2.53}$$

Hence, if $F_{AB}{}^I = 0$, i.e. for vanishing torsion, the target space holonomy group is restricted to be contained in $\mathrm{Sp}(n)$. Note that in general this does not imply that the target space is a hyperkähler manifold, since its dimension is $4n - n_T$. For odd n_T, it cannot even admit a complex structure.

Finally, we check fermion terms of higher order in the supersymmetry variation of the action. Cancellation of the terms proportional to $\lambda\lambda\bar{\lambda}H_\mu$ leads to

$$V_{ab\,\bar{a}\bar{b}}\,\bar{f}^{I\bar{b}}{}_i = -h_{c\bar{a}}\left(\gamma_{ia}^A\,\mathcal{D}_A\,\Gamma^{Ic}{}_b - g_{Jia}\left[\Gamma^I\,,\,\Gamma^J\right]^c{}_b - 2g_{Jid}\,\Gamma^{Id}{}_b\Gamma^{Jc}{}_a\right). \tag{2.54}$$

This constraint has a symmetric and antisymmetric part in a, b. Vanishing of the terms proportional to $\lambda\lambda\bar{\lambda}\partial\phi$ requires

$$V_{ab\,\bar{a}\bar{b}}\,\bar{W}_{Ai}^{\bar{b}} = h_{c\bar{a}}\left(\gamma_{ia}^B\,\mathcal{R}_{AB}{}^c{}_b - 2\Gamma^{Ic}{}_a\,\mathcal{D}_A\,g_{Iib} + g_{Iia}\,\mathcal{D}_A\Gamma^{Ic}{}_b\right), \tag{2.55}$$

which also has a symmetric and antisymmetric part. One can solve this for the curvature,

$$\mathcal{R}_{AB}{}^a{}_b = h^{\bar{a}a}V_{bc\,\bar{a}\bar{b}}\,\bar{W}_{Ai}^{\bar{b}}W_B^{ci} - 2h_{c\bar{b}}\,\bar{W}_{Ai}^{\bar{b}}W_B^{di}\,M_{IJ}\,\Gamma^{Ia}{}_d\,\Gamma^{Jc}{}_b\,. \tag{2.56}$$

Contracting these two constraints with $\bar{g}_{I\bar{e}}^j$ and $\bar{\gamma}^j{}_{\bar{e}}^A$ finally leads to an expression for the four-fermi tensor. Using (2.50) and (2.42), we find

$$\begin{aligned} V_{ab\,\bar{a}\bar{b}}\,\delta_i^j \;=\; & h_{c\bar{a}}\Big(\bar{\gamma}^j{}_{\bar{b}}^A\,\gamma_{ia}^B\,\mathcal{R}_{AB}{}^c{}_b + 2\,\delta_i^j\,M_{IJ}\,\Gamma^{Ic}{}_a\Gamma^{Jd}{}_b\,h_{d\bar{b}} + \\ & + \left(g_{Iia}\bar{\gamma}^j{}_{\bar{b}}^A - \bar{g}_{I\bar{b}}^j\gamma_{ia}^A\right)\mathcal{D}_A\Gamma^{Ic}{}_b - g_{Iia}\,\bar{g}_{J\bar{b}}^j\,\left[\Gamma^I,\,\Gamma^J\right]^c{}_b\Big)\,. \end{aligned} \tag{2.57}$$

To end this section, we demonstrate the closure of the supersymmetry algebra on the tensors. This is more complicated, since the commutator only closes modulo gauge transformations and equations of motion. The latter property is rather unusual: Recall that a supersymmetry commutator is at most linear in derivatives, whereas bosonic equations of motion are second-order differential equations. This is why usually one only finds fermionic field equations, in the commutators evaluated on the fermions. For tensors, a second derivative can be produced by introducing an explicit spacetime coordinate dependence according to the identity (which is valid for any vector, not just for the dual field strength of some tensor)

$$H_{[\mu}\,\sigma_{\nu]} = \sigma^\rho x_\rho\,\partial_{[\mu}H_{\nu]} - \partial_{[\mu}\left(H_{\nu]}\sigma^\rho x_\rho\right),$$

as was first noticed in [10] (see also [21] for a deeper explanation of the explicit coordinate dependence). The first term on the right-hand side is proportional to the (linearized) field equation of a tensor, while the second term has the form of a gauge transformation.

Making use of this identity, we find after some algebra

$$[\,\delta_\epsilon\,,\,\delta_{\epsilon'}\,]B_{\mu\nu I} = -a^\rho\partial_\rho\, B_{\mu\nu I} + 2\,\partial_{[\mu}\Lambda_{\nu]I} + E_{IJ}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\delta S_{\mathrm{T}}}{\delta B_{\rho\sigma J}}\,, \tag{2.58}$$

where

$$\begin{aligned} a^\rho &= \mathrm{i}\left(\epsilon^i\sigma^\rho\bar\epsilon'_i - \epsilon'^i\sigma^\rho\bar\epsilon_i\right) \\ \Lambda_{\nu I} &= a^\rho B_{\rho\nu I} + \mathrm{i}\left(\epsilon^i\sigma^\rho\bar\epsilon'_j - \epsilon'^i\sigma^\rho\bar\epsilon_j\right)x_\rho\,\vec\tau^{\,j}{}_i\,\cdot \\ &\qquad \cdot\left[\left(\vec J_{IA} - \vec J_{IJ}A^J_A\right)\partial_\nu\phi^A + \vec J_{IJ}M^{JK}\left(H_{\nu K} + k_{Ka\bar a}\,\lambda^a\sigma_\nu\bar\lambda^{\bar a}\right)\right] \\ E_{IJ} &= \mathrm{i}\left(\epsilon^i\sigma^\rho\bar\epsilon'_j - \epsilon'^i\sigma^\rho\bar\epsilon_j\right)x_\rho\,\vec\tau^{\,j}{}_i\cdot\vec J_{IJ}\,. \end{aligned} \tag{2.59}$$

Note the explicit x-dependence of the gauge transformation parameters $\Lambda_{\nu I}$ and the antisymmetric matrix E_{IJ} multiplying the field equations, whose origin was explained above. E_{IJ} is field-independent (though not constant), for if the components $\vec J_{IJ} = \mathrm{i}h^{\bar a a}g_{Iia}\vec\tau^{\,i}{}_j\bar g^j_{J\bar a}$ of the complex structures (2.8) are independent of the ϕ^I, they cannot depend on the ϕ^A either:

$$\partial_A\vec J_{IJ} = 3\,\partial_{[A}\vec J_{IJ]} = 3\,D_{[A}\vec J_{IJ]} = 0\,. \tag{2.60}$$

3. Local supersymmetry

In this section we consider hypermultiplets coupled to $N = 2$ supergravity with a certain number of commuting target space isometries. This property we then use to dualize the corresponding scalars into tensors, just as we did for rigid supersymmetry. In this way, we find the most general locally $N = 2$ supersymmetric system of scalars and tensors (with at most two derivatives and undeformed gauge transformations).

For local $N = 2$ supersymmetry, the $4n$ real hypermultiplet scalars parametrize a quaternionic manifold [6]. The holonomy group of such manifolds is contained in $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$, with a nonvanishing $\mathrm{Sp}(1)$ connection.

We denote the supergravity multiplet components by $e_\mu{}^m$, A_μ and ψ^i_μ. The lagrangian then reads[8]

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{H}} = &-\frac{1}{2\kappa^2}\,R(e,\omega) + \varepsilon^{\mu\nu\rho\sigma}\left(D_\mu\psi^i_\nu\sigma_\rho\bar\psi_{\sigma i} + \psi^i_\sigma\sigma_\rho D_\mu\bar\psi_{\nu i}\right) - \frac{1}{4}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \\ &-\frac{\kappa}{2\sqrt2}\left(\tilde{\mathcal{F}}^{\mu\nu} + \tilde F^{\mu\nu}\right)\left(\psi^i_\mu\psi_{\nu i} + \bar\psi^i_\mu\bar\psi_{\nu i}\right) - \frac{\mathrm{i}}{2}\,h_{a\bar a}\left(\lambda^a\sigma^\mu D_\mu\bar\lambda^{\bar a} - D_\mu\lambda^a\sigma^\mu\bar\lambda^{\bar a}\right) - \\ &-\frac12\,G_{\hat A\hat B}\,\hat D^\mu\phi^{\hat A}\,\hat D_\mu\phi^{\hat B} + \kappa\,G_{\hat A\hat B}\left(\hat D_\mu\phi^{\hat A} + \partial_\mu\phi^{\hat A}\right)\left(\gamma^{\hat B}_{ia}\,\lambda^a\sigma^{\mu\nu}\psi^i_\nu + \mathrm{c.c.}\right) - \\ &-\frac{\mathrm{i}\kappa}{2\sqrt2}\,\mathcal{F}_{\mu\nu}\left(\mathcal{E}_{ab}\,\lambda^a\sigma^{\mu\nu}\lambda^b - \mathrm{c.c.}\right) - \frac{\kappa^2}{8}\left(\mathcal{E}_{ac}\mathcal{E}_{bd}\,\lambda^a\lambda^b\,\lambda^c\lambda^d + \mathrm{c.c.}\right) + \\ &+\frac14\,W_{ab\,\bar a\bar b}\,\lambda^a\lambda^b\,\bar\lambda^{\bar a}\bar\lambda^{\bar b}\,. \end{aligned} \tag{3.1}$$

[8]We define dual field strengths $\tilde F^{\mu\nu} = \frac12\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $H^\mu = \frac12\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$ as tensors, where $\varepsilon^{0123} = e^{-1}$.

JHEP04(2003)042

Here, the supercovariant field strength and supercovariant derivative are given by

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= F_{\mu\nu} + \sqrt{2}\,\mathrm{i}\kappa\left(\psi^i_\mu\psi_{\nu i} - \bar\psi^i_\mu\bar\psi_{\nu i}\right)\\ \hat D_\mu\phi^{\hat A} &= \partial_\mu\phi^{\hat A} - \kappa\left(\gamma^{\hat A}_{ia}\,\psi^i_\mu\lambda^a + \bar\gamma^{i\hat A}{}_{\bar a}\,\bar\psi_{\mu i}\bar\lambda^{\bar a}\right),\end{aligned} \tag{3.2}$$

while

$$D_\mu\lambda^a = \nabla_\mu\lambda^a + \partial_\mu\phi^{\hat A}\,\Omega_{\hat A}{}^a{}_b\,\lambda^b\,, \qquad D_\mu\psi^i_\nu = \nabla_\mu\psi^i_\nu + \partial_\mu\phi^{\hat A}\,\Omega_{\hat A}{}^i{}_j\,\psi^j_\nu \tag{3.3}$$

are Lorentz and $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$-covariant derivatives. Notice that now the inverse vielbeins $\gamma^{\hat A}_{ia}$ are appearing explicitly in the lagrangian. $\gamma^{\hat A}_{ia}$ and $V^{ai}_{\hat A}$ are subject to the same algebraic relations as for rigid supersymmetry, with the difference that they are now covariantly constant with respect to the Levi-Civita and $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ connections. This leads to integrability relations which decompose the Riemann curvature into an $\mathrm{Sp}(n)$ and an $\mathrm{Sp}(1)$ part. The quaternionic geometry relates the curvature of the the $\mathrm{Sp}(1)$ connection $\Omega_{\hat A}{}^i{}_j$ to the quaternionic two-forms, as it is expressed by the relation

$$R_{\hat A\hat B}{}^i{}_j = -\frac{\mathrm{i}}{2}\kappa^2\,\vec J_{\hat A\hat B}\cdot\vec\tau^{\,i}{}_j = -\kappa^2\,h_{a\bar a}\,V^{ai}_{[\hat A}\,\bar V^{\bar a}_{\hat B]j}\,. \tag{3.4}$$

These facts imply that the scalar fields span a negatively curved quaternionic manifold [6].

The four-fermi tensor $W_{ab\bar a\bar b}$ in (3.1) is still given by (2.9). That there is no difference with rigid supersymmetry can be seen from the fact that no variations of supergravity fields contribute to the variation of the action proportional to $\lambda\lambda\bar\lambda\partial\phi$. We would like to emphasize however, that the corresponding tensor W_{abcd} is *not* the same as in [6] or as the one derived from appendix B in [20]. In the latter two references, one has a completely symmetric four-index tensor. This tensor differs from ours by a term proportional to $\mathcal{E}_{a(c}\mathcal{E}_{d)b}$, or, for $W_{ab\bar a\bar b}$, by a term proportional to $h_{a(\bar a}h^t_{\bar b)b}$. One can choose to subtract this term from our W and add it as a separate term in the lagrangian (as was done in [6]). To keep formulas shorter, we prefer not to do so, but stress again that both approaches are consistent and equivalent.

The action is invariant under the following local supersymmetry transformations:

$$\begin{aligned}\delta_\epsilon e_\mu{}^m &= \mathrm{i}\kappa\left(\epsilon^i\sigma^m\bar\psi_{\mu i} - \psi^i_\mu\sigma^m\bar\epsilon_i\right)\\ \delta_\epsilon A_\mu &= \sqrt{2}\,\mathrm{i}\left(\epsilon_i\psi^i_\mu + \bar\epsilon^i\bar\psi_{\mu i}\right)\\ \delta_\epsilon\phi^{\hat A} &= \gamma^{\hat A}_{ia}\,\epsilon^i\lambda^a + \bar\gamma^{i\hat A}{}_{\bar a}\,\bar\epsilon_i\,\bar\lambda^{\bar a}\\ \delta_\epsilon\lambda^a &= \mathrm{i}\hat D_\mu\phi^{\hat A}\,V^{ai}_{\hat A}\,\sigma^\mu\bar\epsilon_i - \delta_\epsilon\phi^{\hat A}\,\Omega_{\hat A}{}^a{}_b\,\lambda^b\\ \delta_\epsilon\psi^i_\mu &= \kappa^{-1}D_\mu\epsilon^i + \frac{1}{2\sqrt2}\left(\mathcal{F}_{\mu\nu} + \mathrm{i}\tilde{\mathcal{F}}_{\mu\nu} + \frac{\mathrm{i}\kappa}{\sqrt2}\,\mathcal{E}_{ab}\,\lambda^a\sigma_{\mu\nu}\lambda^b\right)\varepsilon^{ij}\sigma^\nu\bar\epsilon_j - \delta_\epsilon\phi^{\hat A}\,\Omega_{\hat A}{}^i{}_j\,\psi^j_\mu\,.\end{aligned} \tag{3.5}$$

In order to dualize, we again need a number of commuting isometries. Under such isometries generated by $k^{\hat A}_I$, in general not only the λ^a will transform non-trivially, but also the gravitinos, i.e., we have

$$\delta_\theta\lambda^a + \delta_\theta\phi^{\hat A}\,\Omega_{\hat A}{}^a{}_b\lambda^b = \theta^I t_I{}^a{}_b\,\lambda^b\,, \qquad \delta_\theta\psi^i_\mu + \delta_\theta\phi^{\hat A}\,\Omega_{\hat A}{}^i{}_j\psi^j_\mu = \theta^I t_I{}^i{}_j\,\psi^j_\mu\,. \tag{3.6}$$

The two matrices ${t_I}^a{}_b$ and ${t_I}^i{}_j$ can be determined from the requirement that the change of the vielbein $V_{\hat{A}}^{ai}$ be compensated by a combined $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ transformation,

$$
\begin{aligned}
0 &= \mathcal{L}_{k_I} V_{\hat{A}}^{ai} - \left(t_I - k_I^{\hat{B}} \Omega_{\hat{B}}\right)^a{}_b V_{\hat{A}}^{bi} - \left(t_I - k_I^{\hat{B}} \Omega_{\hat{B}}\right)^i{}_j V_{\hat{A}}^{aj} \\
&= D_{\hat{A}} k_I^{\hat{B}} V_{\hat{B}}^{ai} - {t_I}^a{}_b V_{\hat{A}}^{bi} - {t_I}^i{}_j V_{\hat{A}}^{aj} . \qquad (3.7)
\end{aligned}
$$

Using the tracelessness of the t_I matrices, which follows from the relations ${t_I}^c{}_{[a} \mathcal{E}_{b]c} = {t_I}^k{}_{[i} \varepsilon_{j]k} = 0$, we find

$$
{t_I}^a{}_b = \frac{1}{2} V_{\hat{B}}^{ai} \gamma_{ib}^{\hat{A}} D_{\hat{A}} k_I^{\hat{B}} , \qquad {t_I}^i{}_j = \frac{1}{2n} V_{\hat{B}}^{ai} \gamma_{ja}^{\hat{A}} D_{\hat{A}} k_I^{\hat{B}} . \qquad (3.8)
$$

The matrices ${t_I}^a{}_b$ are the same as for rigid supersymmetry, and the matrices ${t_I}^i{}_j$ are proportional to the quaternionic moment maps of the commuting isometries [22]. In analogy with the rigid case, one can now show that there is an $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ frame such that the fermions do not transform under the isometries and, in the Frobenius basis introduced in (2.12), no geometric quantities depend on ϕ^I. In this basis, we have again that

$$
{t_I}^a{}_b = {\Omega_I}^a{}_b , \qquad {t_I}^i{}_j = {\Omega_I}^i{}_j . \qquad (3.9)
$$

The conserved currents of the shift symmetries $\delta_\theta \phi^I = \theta^I$ are given by

$$
e^{-1} J_I^\mu = G_{I\hat{A}} \hat{D}^\mu \phi^{\hat{A}} - \mathrm{i} h_{b\bar{a}} {\Omega_I}^b{}_a \lambda^a \sigma^\mu \bar{\lambda}^{\bar{a}} - 2\varepsilon^{\mu\nu\rho\sigma} {\Omega_I}^j{}_i \psi_\nu^i \sigma_\rho \bar{\psi}_{\sigma j} - 2\kappa \left(h_{a\bar{a}} \bar{V}_{Ii}^{\bar{a}} \lambda^a \sigma^{\mu\nu} \psi_\nu^i + \mathrm{c.c.}\right) \qquad (3.10)
$$

The dualization procedure now works just as in the rigid case. We first gauge the isometries by making the replacement (2.22) in $\mathcal{L}_{\mathrm{H}}$ and add a Lagrange multiplier term $-e H_I^\mu A_\mu^I$ to the gauged action. The auxiliary gauge fields we then eliminate by their equations of motion. The solution is

$$
-G_{IJ} A^{\mu J} = H_I^\mu - e^{-1} J_I^\mu = \mathcal{H}_I^\mu - G_{I\hat{A}} \hat{D}^\mu \phi^{\hat{A}} + \mathrm{i} h_{b\bar{a}} {\Omega_I}^b{}_a \lambda^a \sigma^\mu \bar{\lambda}^{\bar{a}} , \qquad (3.11)
$$

where we have introduced the supercovariant field strengths of the tensors:

$$
\mathcal{H}_I^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \left[\partial_\nu B_{\rho\sigma I} + 4 {\Omega_I}^i{}_j \psi_\nu^j \sigma_\rho \bar{\psi}_{\sigma i} - 2\mathrm{i}\kappa \left(g_{Iia} \psi_\nu^i \sigma_{\rho\sigma} \lambda^a - \mathrm{c.c.}\right)\right] . \qquad (3.12)
$$

$\mathcal{H}_I^\mu$ has to be supercovariant since all other terms in (3.11) are. This then enables us to immediately derive the supersymmetry transformations of the tensors, without working out the gaugino couplings and computing the compensating transformation of the Lagrange multiplier term, as we had to do in the rigid case. If $\delta_\epsilon \mathcal{H}_I^\mu$ is to contain only undifferentiated parameters ϵ^i, $\bar{\epsilon}_i$, then the transformation of $B_{\rho\sigma I}$ has to precisely cancel the inhomogeneous terms in the transformation of the gravitinos. Thus,

$$
\delta_\epsilon B_{\mu\nu I} = 2\mathrm{i}\, g_{Iia} \epsilon^i \sigma_{\mu\nu} \lambda^a - 4\kappa^{-1} {\Omega_I}^i{}_j \epsilon^j \sigma_{[\mu} \bar{\psi}_{\nu]i} + \mathrm{c.c.} \qquad (3.13)
$$

For the transformation of λ^a in the dual formulation we find the same expression as in the previous section, with supercovariant field strengths in place of ordinary ones:

$$
\begin{aligned}
\delta_\epsilon \lambda^a = {} & \mathrm{i} \hat{D}_\mu \phi^A W_A^{ai} \sigma^\mu \bar{\epsilon}_i + \mathrm{i} \left(\mathcal{H}_I^\mu + k_{Ib\bar{b}} \lambda^b \sigma^\mu \bar{\lambda}^{\bar{b}}\right) f^{Iai} \sigma_\mu \bar{\epsilon}_i - \\
& - \delta_\epsilon \phi^A {\Gamma_A}^a{}_b \lambda^b - \left(g_{Iic} \epsilon^i \lambda^c + \bar{g}_{I\bar{c}}^i \bar{\epsilon}_i \bar{\lambda}^{\bar{c}}\right) \Gamma^{Ia}{}_b \lambda^b . \qquad (3.14)
\end{aligned}
$$

The transformations of the bosonic fields $e_\mu{}^m$, A_μ and ϕ^A do not change, so it remains to give the gravitino transformation law, which dualization turns into

$$\begin{aligned}
\delta_\epsilon\psi_\mu^i &= \kappa^{-1}\mathcal{D}_\mu\epsilon^i + \frac{1}{2\sqrt{2}}\left(\mathcal{F}_{\mu\nu} + \mathrm{i}\tilde{\mathcal{F}}_{\mu\nu} + \frac{\mathrm{i}\kappa}{\sqrt{2}}\,\mathcal{E}_{ab}\,\lambda^a\sigma_{\mu\nu}\lambda^b\right)\varepsilon^{ij}\sigma^\nu\bar{\epsilon}_j + \\
&\quad + \kappa^{-1}\left[\mathcal{H}_{\mu I} + k_{Ia\bar{a}}\,\lambda^a\sigma_\mu\bar{\lambda}^{\bar{a}} + \kappa\left(g_{Ika}\,\psi_\mu^k\lambda^a + \bar{g}_{I\bar{a}}^k\,\bar{\psi}_{\mu k}\bar{\lambda}^{\bar{a}}\right)\right]\Gamma^{Ii}{}_j\,\epsilon^j - \\
&\quad - \delta_\epsilon\phi^A\,\Gamma_A{}^i{}_j\,\psi_\mu^j - \left(g_{Ika}\,\epsilon^k\lambda^a + \bar{g}_{I\bar{b}}^k\,\bar{\epsilon}_k\bar{\lambda}^{\bar{b}}\right)\Gamma^{Ii}{}_j\,\psi_\mu^j\,. \qquad (3.15)
\end{aligned}$$

The coefficient functions g_{Iia}, W_A^{ai}, etc. appearing in the above equations are related to hypermultiplet quantities in the same way as in the rigid case. Hence, they satisfy the same relations (2.42)–(2.46). Moreover, we have the relation

$$\Gamma^{Ii}{}_j = M^{IJ}\Omega_J{}^i{}_j \qquad (3.16)$$

between the coefficients which appear in the supersymmetry transformations of the gravitinos and tensors, respectively.

Upon substitution of A_μ^I into the action we obtain the tensor formulation of the theory:

$$\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{T}} &= -\frac{1}{2\kappa^2}\,R - \frac{1}{4}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}\,\mathcal{G}_{AB}\,\hat{D}^\mu\phi^A\,\hat{D}_\mu\phi^B + \frac{1}{2}\,M^{IJ}\mathcal{H}_I^\mu\,\mathcal{H}_{\mu J} - A_A^I H_I^\mu\,\partial_\mu\phi^A + \\
&\quad + \varepsilon^{\mu\nu\rho\sigma}\left(\mathcal{D}_\mu\psi_\nu^i\sigma_\rho\bar{\psi}_{\sigma i} + \psi_\sigma^i\sigma_\rho\mathcal{D}_\mu\bar{\psi}_{\nu i}\right) - \frac{\mathrm{i}}{2}\,h_{a\bar{a}}\left(\lambda^a\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{\bar{a}} - \mathcal{D}_\mu\lambda^a\sigma^\mu\bar{\lambda}^{\bar{a}}\right) + \\
&\quad + \kappa\,\mathcal{G}_{AB}\left(\hat{D}_\mu\phi^A + \partial_\mu\phi^A\right)\left(\gamma_{ia}^B\,\lambda^a\sigma^{\mu\nu}\psi_\nu^i + \mathrm{c.c.}\right) + \kappa M^{IJ}\mathcal{H}_I^\mu\left(g_{Jia}\,\psi_\mu^i\lambda^a + \mathrm{c.c.}\right) - \\
&\quad - \frac{\kappa}{2\sqrt{2}}\left(\tilde{\mathcal{F}}^{\mu\nu} + \tilde{F}^{\mu\nu}\right)\left(\psi_\mu^i\psi_{\nu i} + \bar{\psi}_\mu^i\bar{\psi}_{\nu i}\right) - \frac{\mathrm{i}\kappa}{2\sqrt{2}}\,\mathcal{F}_{\mu\nu}\left(\mathcal{E}_{ab}\,\lambda^a\sigma^{\mu\nu}\lambda^b - \mathrm{c.c.}\right) + \\
&\quad + M^{IJ}k_{Ja\bar{a}}\,\lambda^a\sigma^\mu\bar{\lambda}^{\bar{a}}\left[\mathcal{H}_{\mu I} + \kappa\left(g_{Iib}\,\psi_\mu^i\lambda^b + \mathrm{c.c.}\right)\right] - \\
&\quad - \kappa^2 M^{IJ}\left(g_{Iia}\,\psi_\mu^i\lambda^a + \mathrm{c.c.}\right)\left(g_{Jjb}\,\lambda^b\sigma^{\mu\nu}\psi_\nu^j + \mathrm{c.c.}\right) - \\
&\quad - \frac{\kappa^2}{8}\left(\mathcal{E}_{ac}\mathcal{E}_{bd}\,\lambda^a\lambda^b\,\lambda^c\lambda^d + \mathrm{c.c.}\right) + \frac{1}{4}\,V_{ab\bar{a}\bar{b}}\,\lambda^a\lambda^b\,\bar{\lambda}^{\bar{a}}\bar{\lambda}^{\bar{b}}\,. \qquad (3.17)
\end{aligned}$$

The covariant derivatives $\mathcal{D}_\mu$ of the fermions contain connections $\Gamma_A{}^a{}_b$ and $\Gamma_A{}^i{}_j$, where the former is given by (2.31) and the latter by

$$\Gamma_A{}^i{}_j = \Omega_A{}^i{}_j - A_A^I\,\Omega_I{}^i{}_j\,. \qquad (3.18)$$

For the four-fermi terms, the tensor $V_{ab\bar{a}\bar{b}}$ is again determined by the relation (2.32), but V_{abcd} is no longer completely symmetric. Note also that the $M^{IJ}\mathcal{H}_I^\mu\,\mathcal{H}_{\mu J}$ term gives rise to a similar correction to the four-gravitino coupling of the hypermultiplet action.

Finally, let us derive a relation for the target space curvature in the tensor formulation. The covariant derivatives of the vielbeins W_A^{ai}, etc. given in (2.47) and (2.50) for rigid supersymmetry now include in addition Sp(1) connection coefficients. In particular, $\mathcal{D}_A W_B^{ai}$ is still given by the right-hand side of (2.47), but we now have

$$\begin{aligned}
\gamma_{ia}^A\,\partial_A M^{IJ} &= -2M^{IK}\left(g_{Kib}\Gamma^{Jb}{}_a + g_{Kja}\Gamma^{Jj}{}_i\right) \\
\mathcal{D}_A\,g_{Iia} &= -M_{IJ}\,\mathcal{G}_{AB}\left(\gamma_{ib}^B\,\Gamma^{Jb}{}_a + \gamma_{ja}^B\,\Gamma^{Jj}{}_i\right). \qquad (3.19)
\end{aligned}$$

These relations further constrain $\Gamma^{Ia}{}_b$ and $\Gamma^{Ii}{}_j$, in a way consistent with (3.8) and (3.9).

By computing the commutator of two covariant derivatives on W_A^{ai}, we derive the following curvature relation,

$$\mathcal{R}_{ABCD} = h_{a\bar{a}}\, \mathcal{R}_{AB}{}^{a}{}_{b}\, W_C^{bi}\, \bar{W}_{Di}^{\bar{a}} + \mathcal{R}_{AB}{}^{i}{}_{j}\, W_C^{aj} h_{a\bar{a}} \bar{W}_{Di}^{\bar{a}} + \frac{1}{2} M_{IJ}\, F_{C[A}{}^{I} F_{B]D}{}^{J}\,. \tag{3.20}$$

Due to covariant constancy of $\mathcal{E}_{ab}$ and ε_{ij} with respect to the connections $\Gamma_A{}^a{}_b$ and $\Gamma_A{}^i{}_j$, the corresponding curvatures take values in $sp(n)$ and $sp(1)$, respectively. We conclude that if $F_{AB}{}^I$ vanishes the holonomy group of the target space is contained in $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. In the next section we present an example with vanishing $F_{AB}{}^I$.

4. The universal hypermultiplet and its dual double-tensor multiplet

In order to illustrate the results of the previous section, let us now apply the dualization to the example of the universal hypermultiplet coupled to $N = 2$ supergravity. This multiplet arises in Calabi-Yau threefold compactifications of type-II supergravities, and contains four real scalar fields that parametrize the homogeneous quaternion-Kähler target space $\mathrm{SU}(1,2)/\,\mathrm{U}(2)$ [23]. Out of the isometry group $\mathrm{SU}(1,2)$, we may pick two commuting $\mathrm{U}(1)$ isometries that may be used to dualize one or two scalars into tensors. In fact, in the case of type-IIA the universal hypermultiplet arises after dualization of a tensor multiplet, while for type-IIB it follows from a double-tensor multiplet.

In this section we shall reverse the latter dualization and derive the double-tensor multiplet by dualizing the pseudoscalars (i.e., from a IIB perspective, the axion and one of the RR scalars) in the universal hypermultiplet, according to the procedure of section 3. In this way, we obtain the fermionic terms in the action and the supersymmetry rules for the double-tensor multiplet, which were previously unknown. We should mention, however, that in the framework of compactified type-IIB supergravity the double-tensor multiplet is accompanied by $h_{1,1} > 0$ tensor multiplets, and it is inconsistent to truncate them. The case of the pure double-tensor multiplet, as presented here, should be understood as the mirror theory of type-IIA supergravity on a rigid ($h_{1,2} = 0$) Calabi-Yau, where no tensor multiplets occur.

We parametrize the target space of the universal hypermultiplet by four real scalars $\phi^{\hat{A}} = (\phi,\, \chi, \varphi, \sigma)$. Its metric is given in terms of the matrix-valued vierbein 1-form[9]

$$\mathrm{d}\phi^{\hat{A}}\, V_{\hat{A}}^{ai} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{-\phi/2}(\mathrm{d}\chi - \mathrm{i}\,\mathrm{d}\varphi) & \mathrm{d}\phi + \mathrm{i}\,\mathrm{e}^{-\phi}(\mathrm{d}\sigma + \chi \mathrm{d}\varphi) \\ -\mathrm{d}\phi + \mathrm{i}\,\mathrm{e}^{-\phi}(\mathrm{d}\sigma + \chi\mathrm{d}\varphi) & \mathrm{e}^{-\phi/2}(\mathrm{d}\chi + \mathrm{i}\,\mathrm{d}\varphi) \end{pmatrix}, \tag{4.1}$$

as $G = \mathrm{tr}\,(V \otimes V^\dagger)$. The bosonic part of the action then reads explicitly

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{UH}} = {} & -R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,\mathrm{e}^{-\phi}\left(\partial^\mu\chi\,\partial_\mu\chi + \partial^\mu\varphi\,\partial_\mu\varphi\right) - \\ & - \frac{1}{2}\,\mathrm{e}^{-2\phi}\left(\partial_\mu\sigma + \chi\partial_\mu\varphi\right)^2 + \dots\,. \end{aligned} \tag{4.2}$$

[9]In this section we choose constant tensors $h_{a\bar{a}} = \delta_{a\bar{a}}$, $\varepsilon_{12} = \mathcal{E}_{12} = -1$. Furthermore, we set $\kappa^{-1} = \sqrt{2}$ and rescale the transformation parameters $\epsilon^i \to \sqrt{2}\,\epsilon^i$.

The global $\mathrm{SU}(1,2)$ isometry group has an obvious $\mathrm{U}(1)\times\mathrm{U}(1)$ subgroup, which is generated by constant shifts of $\phi^I = (\varphi, \sigma)$ and which does not act on the other scalars $\phi^A = (\phi, \chi)$. With our parametrization of the vierbein, no quantity will depend on the ϕ^I, so all requirements for the dualization are fulfilled.

Since we consider $n = 1$ hypermultiplets, the holonomy group has two $\mathrm{Sp}(1)$ factors. Their connections follow from covariant constancy of the vierbein (cf. [24]):

$$\begin{aligned} \mathrm{d}\phi^{\hat{A}}\,\Omega_{\hat{A}}{}^a{}_b &= \frac{3\mathrm{i}}{4}\,\mathrm{e}^{-\phi}\begin{pmatrix} -(\mathrm{d}\sigma+\chi\mathrm{d}\varphi) & 0 \\ 0 & (\mathrm{d}\sigma+\chi\mathrm{d}\varphi)\end{pmatrix}, \\ \mathrm{d}\phi^{\hat{A}}\,\Omega_{\hat{A}}{}^i{}_j &= \frac{\mathrm{i}}{4}\,\mathrm{e}^{-\phi}\begin{pmatrix} (\mathrm{d}\sigma+\chi\mathrm{d}\varphi) & 2\,\mathrm{e}^{\phi/2}(\mathrm{d}\varphi+\mathrm{i}\,\mathrm{d}\chi) \\ 2\,\mathrm{e}^{\phi/2}(\mathrm{d}\varphi-\mathrm{i}\,\mathrm{d}\chi) & -(\mathrm{d}\sigma+\chi\mathrm{d}\varphi)\end{pmatrix}. \end{aligned} \tag{4.3}$$

Using (2.9), which is valid also in the local case, the four-fermi term with coefficient $W_{ab\,\bar{a}\bar{b}}$ can be expressed in terms of the symmetric matrix $\Sigma^{ab} = \lambda^a\lambda^b$ as

$$\begin{aligned} \frac{1}{4}\,W_{ab\,\bar{a}\bar{b}}\,\lambda^a\lambda^b\,\bar{\lambda}^{\bar{a}}\bar{\lambda}^{\bar{b}} &= -\frac{1}{8}\,\mathrm{tr}\left[\gamma^{\hat{A}}\Sigma\,R^t_{\hat{A}\hat{B}}\,h\left(\gamma^{\hat{B}}\Sigma\right)^\dagger\right] = \frac{3}{16}\,\mathrm{tr}\,\left[\tau^3\Sigma\tau^3\bar{\Sigma}\right] \\ &= \frac{3}{16}\left(\lambda^1\lambda^1\,\bar{\lambda}^1\bar{\lambda}^1 - 2\,\lambda^1\lambda^2\,\bar{\lambda}^1\bar{\lambda}^2 + \lambda^2\lambda^2\,\bar{\lambda}^2\bar{\lambda}^2\right). \end{aligned} \tag{4.4}$$

We now dualize the scalars ϕ^I into two tensors $B_{\mu\nu I}$ along the lines in the previous sections. The metrics of the double-tensor multiplet follow from (2.28) and read

$$M^{IJ} = \mathrm{e}^{\phi}\begin{pmatrix} 1 & -\chi \\ -\chi & \mathrm{e}^{\phi}+\chi^2\end{pmatrix}, \qquad \mathcal{G}_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\phi}\end{pmatrix}, \qquad A^I_A = 0\,. \tag{4.5}$$

The last equality is a consequence of $G_{\hat{A}\hat{B}}$ being block-diagonal, $G_{AI} = 0$, which simplifies the dualization considerably. For the scalar zweibeins in the dual formulation we find

$$\gamma^{\phi}_{ia} = (W^{ai}_{\phi})^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \qquad \gamma^{\chi}_{ia} = \mathrm{e}^{\phi}(W^{ai}_{\chi})^\dagger = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\phi/2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{4.6}$$

while the tensor zweibeins are given by

$$g_{1\,ia} = -\frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{e}^{-\phi}\begin{pmatrix} -\mathrm{e}^{\phi/2} & \chi \\ \chi & \mathrm{e}^{\phi/2}\end{pmatrix}, \qquad g_{2\,ia} = -\frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{e}^{-\phi}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \tag{4.7}$$

(where $I = 1$ refers to φ and $I = 2$ to σ) and

$$f^{1\,ai} = \frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{e}^{\phi/2}\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \qquad f^{2\,ai} = \frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{e}^{\phi/2}\begin{pmatrix} \chi & \mathrm{e}^{\phi/2} \\ \mathrm{e}^{\phi/2} & -\chi\end{pmatrix}. \tag{4.8}$$

One may check that these quantities satisfy the relations (2.42)–(2.46). In particular, since γ^{χ}_{ia} and W^{ai}_{χ} are both real and proportional to the unit matrix, the off-diagonal terms in (2.43) and (2.46) imply that both g_{Iia} and f^{Iai} must be antihermitean and traceless, which is clearly the case.

The target space connections for the double-tensor multiplet are particularly simple:

$$\Gamma_A{}^a{}_b = 0\,, \qquad \Gamma_\phi{}^i{}_j = 0\,, \qquad \Gamma_\chi{}^i{}_j = \frac{1}{2}\,\mathrm{e}^{-\phi/2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{4.9}$$

Since $A_A^I = 0$, the scalar zweibeins W_A^{ai}, γ_{ia}^A are covariantly constant with respect to these connections. The other quantities that derive from the connections of the hypermultiplet are the gravitino coefficients in the supersymmetry transformations of the tensors (3.13)

$$\Omega_1{}^i{}_j = \frac{\mathrm{i}}{4}\,\mathrm{e}^{-\phi}\begin{pmatrix}\chi & 2\,\mathrm{e}^{\phi/2}\\ 2\,\mathrm{e}^{\phi/2} & -\chi\end{pmatrix}, \qquad \Omega_2{}^i{}_j = \frac{\mathrm{i}}{4}\,\mathrm{e}^{-\phi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{4.10}$$

and the coefficients in the transformations of the fermions

$$\begin{aligned}
\Gamma^{1\,a}{}_b &= 0\,, \qquad \Gamma^{2\,a}{}_b = -\frac{3\mathrm{i}}{4}\,\mathrm{e}^{\phi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\
\Gamma^{1\,i}{}_j &= \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi/2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \Gamma^{2\,i}{}_j = -\frac{\mathrm{i}}{4}\,\mathrm{e}^{\phi/2}\begin{pmatrix}-\mathrm{e}^{\phi/2} & 2\chi\\ 2\chi & \mathrm{e}^{\phi/2}\end{pmatrix}.
\end{aligned} \tag{4.11}$$

Just as in the universal hypermultiplet, the four-λ terms come with field-independent coefficients,

$$\frac{1}{4}\,V_{ab\,\bar a\bar b}\,\lambda^a\lambda^b\,\bar\lambda^{\bar a}\bar\lambda^{\bar b} = -\frac{3}{8}\left(\lambda^1\lambda^1\,\bar\lambda^1\bar\lambda^1 - 2\,\lambda^1\lambda^2\,\bar\lambda^1\bar\lambda^2 + \lambda^2\lambda^2\,\bar\lambda^2\bar\lambda^2\right). \tag{4.12}$$

We now have determined all ingredients needed to write down the complete action and supersymmetry transformations for the double-tensor multiplet coupled to supergravity. We refrain from actually doing so, however, since for applications of this result it is advantageous to keep the vielbeins and connection coefficients in matrix form as given above and not to explicitly perform the sums over the $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ indices. Let us just display the linearized supersymmetry transformations of the fermions, as these are the most relevant ones in studying BPS solutions to the theory:

$$\delta_\epsilon\begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix} = \begin{pmatrix}\mathrm{i}\mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{e}^{\phi/2}\hat H_{\mu 1} & \mathrm{i}\partial_\nu\phi - \mathrm{e}^{\phi}H_{\nu 2}\\ -\mathrm{i}\partial_\mu\phi - \mathrm{e}^{\phi}H_{\mu 2} & \mathrm{i}\mathrm{e}^{-\phi/2}\partial_\nu\chi - \mathrm{e}^{\phi/2}\hat H_{\nu 1}\end{pmatrix}\begin{pmatrix}\sigma^\mu\bar\epsilon_1\\ \sigma^\nu\bar\epsilon_2\end{pmatrix} + \dots\,, \tag{4.13}$$

for the hyperinos, and

$$\begin{aligned}
\delta_\epsilon\begin{pmatrix}\psi_\mu^1\\ \psi_\mu^2\end{pmatrix} &= \begin{pmatrix}2\nabla_\mu + \tfrac{\mathrm{i}}{2}\mathrm{e}^{\phi}H_{\mu 2} & -\mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{i}\mathrm{e}^{\phi/2}\hat H_{\mu 1}\\ \mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{i}\mathrm{e}^{\phi/2}\hat H_{\mu 1} & 2\nabla_\mu - \tfrac{\mathrm{i}}{2}\mathrm{e}^{\phi}H_{\mu 2}\end{pmatrix}\begin{pmatrix}\epsilon^1\\ \epsilon^2\end{pmatrix}\\
&\quad + \frac{1}{2}\left(F_{\mu\nu} + \mathrm{i}\tilde F_{\mu\nu}\right)\begin{pmatrix}\sigma^\nu\bar\epsilon_2\\ -\sigma^\nu\bar\epsilon_1\end{pmatrix} + \dots\,,
\end{aligned} \tag{4.14}$$

for the gravitinos.

Note that H_1^μ always appears in the combination $\hat H_1^\mu \equiv H_1^\mu - \chi H_2^\mu$ in the transformations. This is due to a global symmetry of the double-tensor multiplet, which acts on the scalars by a constant shift of χ and on the tensors such that $\hat{H}_1^\mu$ and H_2^μ are invariant [2] (in fact, this applies to the full supercovariant field strengths $\mathcal{H}_I^\mu$). There is another global symmetry, which acts on $\tau \equiv \chi + 2\mathrm{i}\,\mathrm{e}^{\phi/2}$ by $\tau \to \mathrm{e}^{\alpha}\tau$ and on the tensors by $B_{\mu\nu I} \to \mathrm{e}^{-I\alpha} B_{\mu\nu I}$ (no sum). These symmetries do not act on the λ^a thanks to our choosing $h_{a\bar{a}}$ and $\mathcal{E}_{ab}$ constant, and they explain why the four-fermi term (4.12) is independent of the scalars.

Acknowledgments

UT is supported by FWF Grant P15553-N08 and was supported by DFG Grant TH 786/1-2 during the initial stage of this work.

References

[1] G.W. Gibbons, M.B. Green and M.J. Perry, *Instantons and seven-branes in type-IIB superstring theory*, *Phys. Lett.* **B 370** (1996) 37 [hep-th/9511080];
M.B. Green and M. Gutperle, *Effects of D-instantons*, *Nucl. Phys.* **B 498** (1997) 195 [hep-th/9701093].

[2] U. Theis and S. Vandoren, *Instantons in the double-tensor multiplet*, *J. High Energy Phys.* **09** (2002) 059 [hep-th/0208145].

[3] N. Berkovits and W. Siegel, *Superspace effective actions for 4D compactifications of heterotic and type-II superstrings*, *Nucl. Phys.* **B 462** (1996) 213 [hep-th/9510106].

[4] S. Ferrara and S. Sabharwal, *Dimensional reduction of type-II superstrings*, *Class. and Quant. Grav.* **6** (1989) L77;
M. Bodner, A.C. Cadavid and S. Ferrara, *(2,2) vacuum configurations for type-IIA superstrings: $N = 2$ supergravity lagrangians and algebraic geometry*, *Class. and Quant. Grav.* **8** (1991) 789.

[5] M. Bodner and A.C. Cadavid, *Dimensional reduction of type-IIB supergravity and exceptional quaternionic manifolds*, *Class. and Quant. Grav.* **7** (1990) 829;
R. Böhm, H. Günther, C. Herrmann and J. Louis, *Compactification of type-IIB string theory on Calabi-Yau threefolds*, *Nucl. Phys.* **B 569** (2000) 229 [hep-th/9908007].

[6] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, *Nucl. Phys.* **B 222** (1983) 1.

[7] B. de Wit, R. Philippe and A. Van Proeyen, *The improved tensor multiplet in $N = 2$ supergravity*, *Nucl. Phys.* **B 219** (1983) 143.

[8] B. de Wit, M. Roček and S. Vandoren, *Hypermultiplets, hyper-Kähler cones and quaternion-Kähler geometry*, *J. High Energy Phys.* **02** (2001) 039 [hep-th/0101161].

[9] V.I. Ogievetsky and I.V. Polubarinov, *The notoph and its possible interactions, Yad. Fiz.* **4** (1966) 216, [*Sov. J. Nucl. Phys.* **4** (1967) 156];
D.Z. Freedman and P.K. Townsend, *Antisymmetric tensor gauge theories and nonlinear sigma models*, *Nucl. Phys.* **B 177** (1981) 282.

[10] F. Brandt, *New $N = 2$ supersymmetric gauge theories: the double tensor multiplet and its interactions*, *Nucl. Phys.* **B 587** (2000) 543 [hep-th/0005086].

[11] L. Alvarez-Gaumé and D. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model, Commun. Math. Phys.* **80** (1981) 443

[12] J. De Jaegher, B. de Wit, B. Kleijn and S. Vandoren, *Special geometry in hypermultiplets*, *Nucl. Phys.* **B 514** (1998) 553 [hep-th/9707262].

[13] B. de Wit, B. Kleijn and S. Vandoren, *Rigid $N = 2$ superconformal hypermultiplets*, in *Supersymmetries and quantum symmetries*, proceeding Int. Sem. Dubna 1997, J. Wess and E.A. Ivanov eds., *Lecture Notes in Physics*, **524**, Springer, 1999 [hep-th/9808160];
B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, *Nucl. Phys.* **B 568** (2000) 475 [hep-th/9909228].

[14] J. Wess and J. Bagger, *Supersymmetry and supergravity*, *Princeton Series in Physics*, Princeton University Press, Princeton, 1983.

[15] U. Lindström and M. Roček, *Scalar tensor duality and $N = 1$, $N = 2$ nonlinear sigma models*, *Nucl. Phys.* **B 222** (1983) 285;
N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkähler metrics and supersymmetry, Commun. Math. Phys.* **108** (1987) 535.

[16] L. Anguelova, M. Roček and S. Vandoren, *Hyper-Kähler cones and orthogonal Wolf spaces*, *J. High Energy Phys.* **05** (2002) 064 [hep-th/0202149].

[17] P.S. Howe, A. Karlhede, U. Lindström and M. Roček, *The geometry of duality*, *Phys. Lett.* **B 168** (1986) 89.

[18] M. Roček and E. Verlinde, *Duality, quotients, and currents, Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053]9119953

[19] H. Pederson and Y-S. Poon, *Hyper-Kahler metrics and generalization of the Bogomology equations, Commun. Math. Phys.* **117** (1988) 569.

[20] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, R. Halbersma, S. Vandoren and A. Van Proeyen, *Superconformal $N = 2$, $D = 5$ matter with and without actions*, *J. High Energy Phys.* **10** (2002) 045 [hep-th/0205230].

[21] F. Brandt, *Hidden symmetries of supersymmetric p-form gauge theories*, *Phys. Lett.* **B 494** (2000) 325 [hep-th/0009133].

[22] K. Galicki, *A generalization of the momentum mapping construction for quaternionic manifolds, Commun. Math. Phys.* **108** (1987) 117;
L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *N=2 supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032].

[23] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475;
S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type-II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.

[24] A. Strominger, *Loop corrections to the universal hypermultiplet*, *Phys. Lett.* **B 421** (1998) 139 [hep-th/9706195].